CHAPTER 13

MATRICES

In orderly disorder they
Wait coldly columned, dead, prosaic.
Poet, breathe on them and pray
They burn with life in your mosaic.
J. LUZZATO

A graph is completely determined by either its adjacencies or its incidences. This information can be conveniently stated in matrix form. Indeed, with a given graph, adequately labeled, there are associated several matrices, including the adjacency matrix, incidence matrix, cycle matrix, and cocycle matrix. It is often possible to make use of these matrices in order to identify certain properties of a graph. The classic theorem on graphs and matrices is the Matrix-Tree Theorem, which gives the number of spanning trees in any labeled graph. The matroids associated with the cycle and cocycle matrices of a graph are discussed.

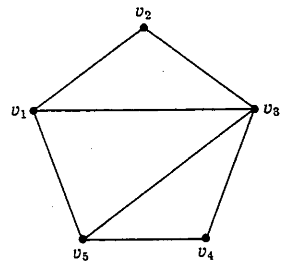
THE ADJACENCY MATRIX

The *adjacency matrix* $A = [a_{ij}]$ *of a labeled graph* G with p points is the $p \times p$ matrix in which $a_{ij} = 1$ if v_i is adjacent with v_j and $a_{ij} = 0$ otherwise. Thus there is a one-to-one correspondence between labeled graphs with p points and $p \times p$ symmetric binary matrices with zero diagonal.

Figure 13.1 shows a labeled graph G and its adjacency matrix A. One immediate observation is that the row sums of A are the degrees of the points of G. In general, because of the correspondence between graphs and matrices, any graph-theoretic concept is reflected in the adjacency matrix. For example, recall from Chapter 2 that a graph G is connected if and only if there is no partition $V = V_1 \cup V_2$ of the points of G such that no line joins a point of V_1 with a point of V_2. In matrix terms we may say that G is connected if and only if there is no labeling of the points of G such that its adjacency matrix has the reduced form

$$A = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix},$$

G:



$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Fig. 13.1. A labeled graph and its adjacency matrix.

where A_{11} and A_{22} are square. If A_1 and A_2 are adjacency matrices which correspond to two different labelings of the same graph G, then for some permutation matrix P, $A_1 = P^{-1}A_2P$. Sometimes a labeling is irrelevant, as in the following results which interpret the entries of the powers of the adjacency matrix.

Theorem 13.1 Let G be a labeled graph with adjacency matrix A. Then the i, j entry of A^n is the number of walks of length n from v_i to v_j.

Corollary 13.1(a) For $i \neq j$, the i, j entry of A^2 is the number of paths of length 2 from v_i to v_j. The i, i entry of A^2 is the degree of v_i and that of A^3 is twice the number of triangles containing v_i.

Corollary 13.1(b) If G is connected, the distance between v_i and v_j for $i \neq j$ is the least integer n for which the i, j entry of A^n is nonzero.

The *adjacency matrix of a labeled digraph D* is defined similarly: $A = A(D) = [a_{ij}]$ has $a_{ij} = 1$ if arc v_iv_j is in D and is 0 otherwise. Thus $A(D)$ is not necessarily symmetric. Some results for digraphs using $A(D)$ will be given in Chapter 16. By definition of $A(D)$, the adjacency matrix of a given graph can also be regarded as that of a symmetric digraph. We now apply this observation to investigate the determinant of the adjacency matrix of a graph, following [H27].

A *linear subgraph of a digraph D* is a spanning subgraph in which each point has indegree one and outdegree one. Thus it consists of a disjoint spanning collection of directed cycles.

Theorem 13.2 If D is a digraph whose linear subgraphs are D_i, $i = 1, \cdots, n$, and D_i has e_i even cycles, then

$$\det A(D) = \sum_{i=1}^{n} (-1)^{e_i}.$$

Every graph G is associated with that digraph D with arcs v_iv_j and v_jv_i whenever v_i and v_j are adjacent in G. Under this correspondence, each linear subgraph of D yields a spanning subgraph of G consisting of a point disjoint collection of lines and cycles, which is called a *linear subgraph of a graph.*

Those components of a linear subgraph of G which are lines correspond to the 2-cycles in the linear subgraph of D in a one-to-one fashion, but those components which are cycles of G correspond to two directed cycles in D. Since $A(G) = A(D)$ when G and D are related as above, the determinant of $A(G)$ can be calculated.

Corollary 13.2(a) If G is a graph whose linear subgraphs are G_i, $i = 1, \cdots, n$, where G_i has e_i even components and c_i cycles, then

$$\det A(G) = \sum_{i=1}^{n} (-1)^{e_i} 2^{c_i}.$$

THE INCIDENCE MATRIX

A second matrix, associated with a graph G in which the points and lines are labeled, is the *incidence matrix* $B = [b_{ij}]$. This $p \times q$ matrix has $b_{ij} = 1$ if v_i and x_j are incident and $b_{ij} = 0$ otherwise. As with the adjacency matrix, the incidence matrix determines G up to isomorphism. In fact any $p - 1$ rows of B determine G since each row is the sum of all the others modulo 2.

The next theorem relates the adjacency matrix of the line graph of G to the incidence matrix of G. We denote by B^T the transpose of matrix B.

Theorem 13.3 For any (p, q) graph G with incidence matrix B,

$$A(L(G)) = B^T B - 2I_q$$

Let M denote the matrix obtained from $-A$ by replacing the ith diagonal entry by $\deg v_i$. The following theorem is contained in the pioneering work of Kirchhoff [K7].

Theorem 13.4 (Matrix-Tree Theorem) Let G be a connected labeled graph with adjacency matrix A. Then all cofactors of the matrix M are equal and their common value is the number of spanning trees of G.

Proof. We begin the proof by changing either of the two 1's in each column of the incidence matrix B of G to -1, thereby forming a new matrix E. (We will see in Chapter 16 that this amounts to arbitrarily orienting the lines of G and taking E as the incidence matrix of this oriented graph.)

The i, j entry of EE^T is $e_{i1}e_{j1} + e_{i2}e_{j2} + \cdots + e_{iq}e_{jq}$, which has the value $\deg v_i$ if $i = j$, -1 if v_i and v_j are adjacent, and 0 otherwise. Hence $EE^T = M$.

Consider any submatrix of E consisting of $p - 1$ of its columns. This $p \times (p - 1)$ matrix corresponds to a spanning subgraph H of G having $p - 1$ lines. Remove an arbitrary row, say the kth, from this matrix to obtain a square matrix F of order $p - 1$. We will show that $|\det F|$ is 1 or 0 according as H is or is not a tree. First, if H is not a tree, then because H has p points and $p - 1$ lines, it is disconnected, implying that there is a

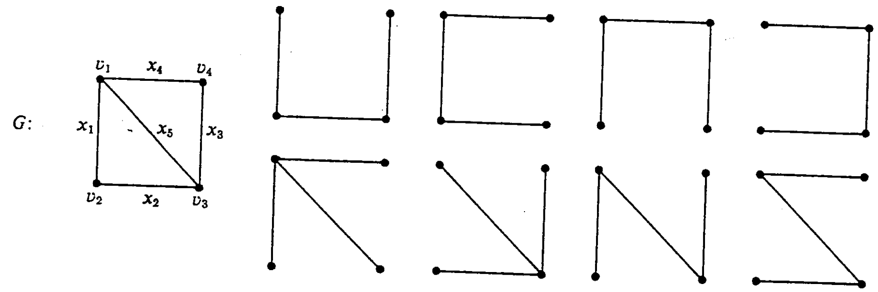


Fig. 13.2. $K_4 - x$ and its spanning trees.

component not containing v_k. Since the rows corresponding to the points of this component are dependent, $\det F = 0$. On the other hand, suppose H is a tree. In this case, we can relabel its lines and points other than v_k as follows: Let $u_1 \neq v_k$ be an endpoint of H (whose existence is guaranteed by Corollary 4.1(a)), and let y_1 be the line incident with it; let $u_2 \neq v_k$ be any endpoint of $H - u_1$ and y_2 its incident line, and so on. This relabeling of the points and lines of H determines a new matrix F' which can be obtained by permuting the rows and columns of F independently. Thus $|\det F'| = |\det F|$. However, F' is lower triangular with every diagonal entry $+1$ or -1; hence, $|\det F| = 1$.

The following algebraic result, usually called the Binet-Cauchy Theorem, will now be very useful.

Lemma 13.4(a) If P and Q are $m \times n$ and $n \times m$ matrices, respectively, with $m \leq n$, then $\det PQ$ is the sum of the products of corresponding major determinants of P and Q.

(A major determinant of P or Q has order m, and the phrase "corresponding major determinants" means that the columns of P in the one determinant are numbered like the rows of Q in the other.)

We apply this lemma to calculate the first principal cofactor of M. Let E_1 be the $(p - 1) \times q$ submatrix obtained from E by striking out its first row. By letting $P = E_1$ and $Q = E_1^T$, we find, from the lemma, that the first principal cofactor of M is the sum of the products of the corresponding major determinants of E_1 and E_1^T. Obviously, the corresponding major determinants have the same value. We have seen that their product is 1 if the columns from E_1 correspond to a spanning tree of G and is 0 otherwise. Thus the sum of these products is exactly the number of spanning trees.

The equality of all the cofactors, both principal and otherwise, holds for every matrix whose row sums and column sums are all zero, completing the proof.

To illustrate the Matrix-Tree Theorem, we consider a labeled graph G taken at random, say $K_4 - x$. This graph, shown in Fig. 13.2, has eight

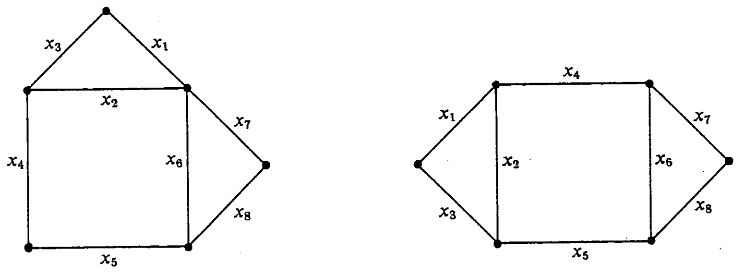


Fig. 13.3. Two graphs with the same cycle matrix.

spanning trees, since the 2,3 cofactor, for example,

$$\text{of}\quad M = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix} \qquad \text{is} \quad -\begin{vmatrix} 3 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & 0 & 2 \end{vmatrix} = 8.$$

The number of labeled trees with p points is easily found by applying the Matrix-Tree Theorem to K_p. Each principal cofactor is the determinant of order $p - 1$:

$$\begin{vmatrix} p-1 & -1 & \cdot\ \cdot\ \cdot & -1 \\ -1 & p-1 & \cdot\ \cdot\ \cdot & -1 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ -1 & -1 & \cdot\ \cdot\ \cdot & p-1 \end{vmatrix}$$

Subtracting the first row from each of the others and adding the last $p - 2$ columns to the first yields an upper triangular matrix whose determinant is p^{p-2}.

Corollary 13.4(a) The number of labeled trees with p points is p^{p-2}.

There appear to be as many different ways of proving this formula as there are independent discoveries thereof. An interesting compilation of such proofs is presented in Moon [M15].

THE CYCLE MATRIX

Let G be a graph whose lines and cycles are labeled. The *cycle matrix* $C = [c_{ij}]$ of G has a row for each cycle and a column for each line with $c_{ij} = 1$ if the ith cycle contains line x_j and $c_{ij} = 0$ otherwise. In contrast to the adjacency and incidence matrices, the cycle matrix does not determine a graph up to isomorphism. Obviously the presence or absence of lines which lie on no cycle is not indicated. Even when such lines are excluded, however, C does not determine G, as is shown by the pair of graphs in Fig. 13.3,

which both have cycles

$$\begin{aligned} Z_1 &= \{x_1, x_2, x_3\} & Z_2 &= \{x_2, x_4, x_5, x_6\} \\ Z_3 &= \{x_6, x_7, x_8\} & Z_4 &= \{x_1, x_3, x_4, x_5, x_6\} \\ Z_5 &= \{x_2, x_4, x_5, x_7, x_8\} & Z_6 &= \{x_1, x_3, x_4, x_5, x_7, x_8\} \end{aligned}$$

and therefore share the cycle matrix

$$C = \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \end{bmatrix} \end{array} \begin{matrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \\ Z_5 \\ Z_6 \end{matrix}$$

The next theorem provides a relationship between the cycle and incidence matrices. In combinatorial topology this result is described by saying that the boundary of the boundary of any chain is zero.

Theorem 13.5 If G has incidence matrix B and cycle matrix C, then

$$CB^T \equiv 0 \pmod 2.$$

Proof. Consider the ith row of C and jth column of B^T, which is the jth row of B. The rth entries in these two rows are both nonzero if and only if x_r is in the ith cycle Z_i and is incident with v_j. If x_r is in Z_i, then v_j is also, but if v_j is in the cycle, then there are two lines of Z_i incident with v_j so that the i, j entry of CB^T is $1 + 1 \equiv 0 \pmod 2$.

Analogous to the cycle matrix, one can define the *cocycle matrix* $C^*(G)$. If G is 2-connected, then each point of G corresponds to the cocycle (minimal cutset) consisting of the lines incident with it. Therefore, the incidence matrix of a block is contained in its cocycle matrix.

Since every row of the incidence matrix B is the sum modulo 2 of the other rows, it is clear that the rank of B is at most $p - 1$. On the other hand, if the rank of B is less than $p - 1$, then there is some set of fewer than p rows whose sum, modulo 2, is zero. But then there can be no line joining a point in the set belonging to those rows and a point not in that set, so G cannot be connected. Thus we have one part of the next theorem. The other parts follow directly from the results in Chapter 4 which give the dimensions of the cycle and cocycle spaces of G.

Theorem 13.6 For a connected graph G, the ranks of the cycle, incidence, and cocycle matrices are $r(C) = q - p + 1$ and $r(B) = r(C^*) = p - 1$.

In view of Theorem 13.6, an important submatrix of the cycle matrix C of a connected graph is given by any $m = q - p + 1$ rows representing a

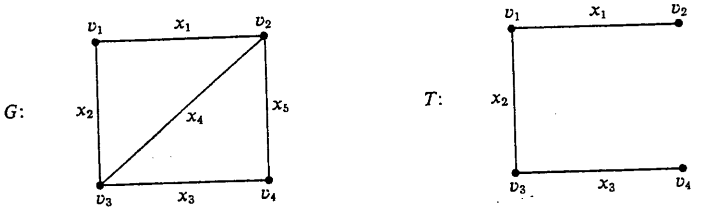


Fig. 13.4. A graph and a spanning tree.

cycle basis. Each such reduced matrix $C_0(G)$ is an $m \times q$ submatrix of C, and similarly a reduced cocyle matrix $C_0^*(G)$ is $m^* \times q$, where $m^* = p - 1$. Then by Theorem 13.5, we have immediately $CC^{*T} \equiv 0 \pmod 2$ and hence also $C_0 C_0^{*T} \equiv 0 \pmod 2$. A reduced incidence matrix B_0 is obtained from B by deletion of the last row. By an earlier remark, no information is lost by so reducing B.

If the cycles and cocycles are chosen in a special way, then the reduced incidence, cycle, and cocycle matrices of a graph have particularly nice forms. Recall from Chapter 4 that any spanning tree T determines a cycle basis and a cocycle basis for G. In particular, if $X_1 = \{x_1, x_2, \cdots, x_{p-1}\}$ is the set of twigs (lines) of T, and $X_2 = \{x_p, x_{p+1}, \cdots, x_q\}$ is the set of its chords, then there is a unique cycle Z_i in $G - X_2 + x_i$, $p \le i \le q$, and a unique cocycle Z_j^* in $G - X_1 + x_j$, $1 \le j \le p - 1$, and these collections of cycles and cocycles form bases for their respective spaces. For example, in the graph G of Fig. 13.4 the cycles and cocycles determined by the particular spanning tree T shown are

$$Z_4 = \{x_1, x_2, x_4\}, \qquad Z_1^* = \{x_1, x_4, x_5\},$$
$$Z_5 = \{x_1, x_2, x_3, x_5\}, \qquad Z_2^* = \{x_2, x_4, x_5\},$$
$$Z_3^* = \{x_3, x_5\}.$$

The reduced matrices, which are determined both by G and the choice of T, are:

$$B_0(G, T) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \overbrace{x_1 \; x_2 \; x_3}^{X_1} \; \overbrace{x_4 \; x_5}^{X_2} \\ \left[\begin{array}{ccc|cc} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{array}\right] \end{array},$$

$$C_0(G, T) = \begin{array}{c} Z_4 \\ Z_5 \end{array} \overbrace{\left[\begin{array}{ccc|cc} 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{array}\right]}^{X_1 \qquad X_2},$$

and

$$C_0^*(G, T) = \begin{array}{c} Z_1^* \\ Z_2^* \\ Z_3^* \end{array} \overbrace{\left[\begin{array}{ccc|cc} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right]}^{X_1 \qquad X_2}.$$

It is easy to see that this is a special case of the following equations (all modulo 2) which hold for any connected graph G and spanning tree T:

$$B_0 = B_0(G, T) = [\overbrace{B_1}^{X_1} \quad \overbrace{B_2}^{X_2}], \qquad C_0 = C_0(G, T) = [\overbrace{C_1}^{X_1} \quad \overbrace{I_m}^{X_2}],$$

and

$$C_0^* = C_0^*(G, T) = [\overbrace{I_{m^*}}^{X_1} \quad \overbrace{C_2^*}^{X_2}],$$

where $C_1^T = B_1^{-1}B_2 = C_2^*$ and $C_0^* = B_1^{-1}B_0 = [I_{m^*} \quad C_1^T]$. It follows from these equations that, given G and T, each of the partitioned matrices B_0, C_0, and C_0^* determines the other two.

Excursion—Matroids Revisited

The cycle and cocycle matrices are particular representations of the cycle matroid and cocycle matroid of a graph, introduced in Chapter 4. A matroid is called *graphical* if it is the cycle matroid of some graph, and *cographical* if it is a cocycle matroid. Tutte [T12] has determined which matroids are graphical or cographical, thereby inadvertently solving a previously open problem in electric network theory.

The smallest example of a matroid which is not graphical or cographical is the self-dual matroid obtained by taking $M = \{1, 2, 3, 4\}$ and the circuits all 3-element subsets of M.

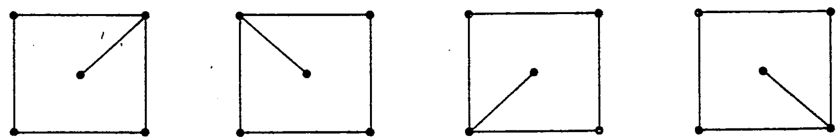


Fig. 13.5. The new circuits in the whirl of W_5.

Another example, Tutte [T19], of a matroid which is not graphical involves the wheel $W_{n+1} = K_1 + C_n$. Its cycle matroid has $n^2 - n + 1$ circuits since there are that many cycles in a wheel. If in this matroid we remove from the collection of circuits the cycle C_n which forms the rim of the wheel, and add to it all of the "spoked rims" (the sets of lines in the subgraphs shown in Fig. 13.5), then it can be shown that the result is a new matroid which is not graphical or cographical. This is called a *whirl* of order n and is generated by n^2 circuits.

Even if a matroid is graphical, it need not be cographical. For example, the cycle matroid of K_5 is not cographical. In fact a matroid is both graphical and cographical if and only if it is the cycle matroid of some planar graph.

EXERCISES

13.1 a) Characterize the adjacency matrix of a bipartite graph.
b) A graph G is bipartite if and only if for all odd n every diagonal entry of A^n is 0.

13.2 Let G be a connected graph with adjacency matrix A. What can be said about A if

a) v_i is a cutpoint?
b) v_iv_j is a bridge?

13.3 If $c_n(G)$ is the number of n-cycles of a graph G with adjacency matrix A, then

a) $c_3(G) = \frac{1}{6}\text{tr}(A^3)$.
b) $c_4(G) = \frac{1}{8}[\text{tr}(A^4) - 2q - 2\sum_{i\neq j} a_{ij}^{(2)}]$.
c) $c_5(G) = \frac{1}{10}[\text{tr}(A^5) - 5\,\text{tr}(A^3) - 5\sum_{i=1}^{p}(\sum_{j=1}^{p} a_{ij} - 2)a_{ii}^{(3)}]$.

(Harary and Manvel [HM1])

13.4 a) If G is a disconnected labeled graph, then every cofactor of M is 0.
b) If G is connected, the number of spanning trees of G is the product of the number of spanning trees of the blocks of G.

(Brooks, Smith, Stone, and Tutte [BSST1])

13.5 Let G be a labeled graph with lines $x_1, x_2, \cdots, x_q$. Define the $p \times p$ matrix $M_x = [m_{ij}]$ by

$$m_{ij} = \begin{cases} -x_k & \text{if } x_k = v_iv_j \\ 0 & \text{if } v_i \text{ and } v_j \text{ are not adjacent} \end{cases} \quad \text{for } i \neq j,$$

$$-m_{ii} = \sum_{n\neq i} m_{in}.$$

By the term of a spanning tree of G is meant the product of its lines. The tree polynomial of G is defined as the sum of the terms of its spanning trees.

The Variable Matrix Tree Theorem asserts that the value of any cofactor of the matrix M_x is the tree polynomial of G.

13.6 Do there exist two different graphs with the same cycle matrix which are smaller than those in Fig. 13.3?

13.7 The "cycle-matroid" and "cocycle-matroid" of a graph do indeed satisfy the first definition of matroid given in Chapter 4.

13.8 Two graphs G_1 and G_2 are *cospectral* if the polynomials $\det(A_1 - tI)$ and $\det(A_2 - tI)$ are equal. There are just two different cospectral graphs with 5 points.

(F. Harary, C. King, and R. C. Read)

13.9 If the eigenvalues of $A(G)$ are distinct, then every nonidentity automorphism of G has order 2. (Mowshowitz [M17])

13.10 Let $f(t)$ be a polynomial of minimum degree (if any) such that every entry of $f(A)$ is 1, where A is the adjacency matrix of G. Then a graph has such a polynomial if and only if it is connected and regular. (Hoffman [H45])

13.11 An *eulerian matroid* has a partition of its set S of elements into circuits.

a) A graphical matroid is eulerian if and only if it is the cycle matroid of an eulerian graph,
b) Not every eulerian matroid is graphical.

13.12 In a *binary* matroid, the intersection of every circuit and cocircuit has even cardinality. Every cocircuit of a binary eulerian matroid has even cardinality. In other words, the dual of a binary eulerian matroid is a "bipartite matroid," defined as expected.

(Welsh [W9])